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TIMOSHENKO EQUATION OF HYPERBOLIC TYPE AND BASIC SINGULARITIES

Ua Показано, що рівняння Тимошенка принципово відрізняється від класичного рівняння Бернулі-Ейлера згинних коливань балки, воно описує поширення збурень зі скінченою швидкістю, що не задовольняє закону суцільності середовища, і у цьому випадку відповідає моделі Косера. Це встановлено із наведеного узагальненого рівняння 6-го порядку гіперболічного типу, яке, як часткові випадки, включає відомі рівняння, а також рівняння Тимошенка без коректуючого коефіцієнта зсуву. Відмічаємо, що підбір цього коректуючого коефіцієнта проводили дослідники на основі точних розв'язків, які отримані з моделі без порушення суцільності середовища. У подальшому було розроблено обчислювальні програми, але із набагато більшими порушеннями суцільності середовища.

Відмічаємо рівняння 6-го порядку, виведене із *n*-мірного евклідового простору, узагальнюючого метод Коші-Пуасона.

Ru Показано, что уравнение Тимошенко принципиально отличается от классического уравнения Бернулли-Эйлера изгибных колебаний балки, оно описывает распространение возмущений с конечной скоростью, что не удовлетворяет закону сплошности среды, и в этом случае соответствует модели Коссера. Это устанавливается из приведенного обобщенного уравнения 6-го порядка гиперболического типа, которое, как частные случаи, включает известные уравнения, а также уравнение Тимошенко без корректирующего коэффициента сдвига.

Отмечается, что подбор этого корректирующего коэффициента проводился исследователями на основе точных решений, полученных из модели без нарушения сплошности среды. В дальнейшем были составлены вычислительные программы с еще большими нарушениями сплошности среды.

Отмечается уравнение 6-го порядка, выведенное из *n*-мерного евклидового пространства, обобщающего метод Коши-Пуассона.

Introduction

The analysis of works [1], [2] shows the singularities of Timoshenko model [3] unlike previous studies. Cauchy [4], Poisson [5] considered the problem of plate bending (beam-strips) based on the dynamic theory of elasticity (elastodynamics).

This method was generalized to *n*-dimensional Euclidean space. On this basis, a generalized 6th-order hyperbolic equation that describes the propagation of perturbations at a finite speed was derived. This equation includes, as special

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cases, the well-known equations and the Timoshenko equation without the shear coefficient thickness correcting. The problem of studying the singularity of the Tymoshenko equation is considered in this article in connection with the violation of the continuity of the elastic medium [2]. Until now, many publications on the Timoshenko equation were presented [3] without taking into account its singularity.

Maxwell [6] was the first who considered the construction of hyperbolic equations describing the propagation of disturbances with a finite velocity and this idea was realized later by Einstein [7]. The publication [8] considered methods of solutions of wave propagation. Cosserat [9] developed the model violated the continuity of medium.

The aim of this article is to study the Timoshenko equation without the shear coefficient thickness correcting.

IBV-problem developing

After considering the problem in n-dimensional Euclidean space, differential equations and boundary conditions can be written as

$$L_1 = \alpha_k \frac{\partial^q u_k}{\partial x^1 \ldots \partial x^n} + F_k = P_k, \quad n = 1, 2, \ldots, p \quad \text{at} \quad \Omega,\ t \geq 0, \tag{1}$$

$$\left(b_k \frac{\partial^q u_k}{\partial x^1 \ldots \partial x^n} + F_k \right)_{x^s = + = h^s} = Q_k^{\pm}, \tag{2}$$

where (1) is a system of k equations of the p first order with k unknown functions, which must be defined as its solutions satisfying the boundary conditions (2) and the initial conditions in the case of the IBV–problem, so that the correct statement of the problem is guaranteed. The first term in (1) is the main part of the operator, the second term remains as a part of the lower order operator. In (2), the term is an operator of lower order than the first term. It is assumed that the coefficients are constants, but they can depend on a small parameter ε.

Following the foregoing, let us consider the construction of degenerate models for the case R^4. This is the problem of elastodynamics for a layer. The mathematical formulation of the corresponding IBV-problem for an elastic isotropic medium in terms of displacements $u_1,\ u_2,\ u_3$ that depend on orthogonal coordinates $x_1,\ x_2,\ x_3$ and time t is represented as follows: find the vector function $\vec{u}(x_1, x_2, x_3, t)$ as a solution of the equations in $\Omega \times [0, T],\ T > 0$

$$\nabla^2 u_k + (1 + \lambda / G)\partial_k (\vec{\nabla} \cdot \vec{u}) + P_k = \partial_{tt} u_k, \quad k = 1, 2, 3, \tag{3}$$

satisfying the boundary conditions

$$\begin{gathered} \sigma_{33}]_{x_3 = \xi/2} = q^+(x_1, x_2, t), \qquad \sigma_{33}]_{x_3 = -\xi/2} = q^-(x_1, x_2, t), \\ \sigma_{3i}]_{x_3 = \xi/2} = p_i^+(x_1, x_2, t), \qquad \sigma_{3i}]_{x_3 = -\xi/2} = p_i^-(x_1, x_2, t), \quad (i = 1, 2) \end{gathered} \tag{4}$$

and initial conditions

$$u_k]_{t=0}=0,\_\_\partial_t u_k]_{t=0}=0,\_\_k=1,2,3. \qquad (5)$$

The components of the displacement vector are represented in the form of power series in x_3

$$u_i(x_1,x_2,x_3,t)=\sum_{\nu=0}^{\infty}u_{i\nu}(x_1,x_2,t)x_3^{\nu}\_,\_\_i=1,2,3. \qquad (6)$$

The functions $u_{i\nu}$ are assumed to be differentiable as many times as required, all the derivatives $u_{i\nu}$ are continuous, and the series (6) converges uniformly. Mass forces P_k are not taken into account in the future.

Dimensionless quantities are introduced using the formulas, taking thickness $2h(m)$, shear wave velocity $c_s(m/s)$, density $\rho(kg/m)$ as the characteristic

$$u_k^*=\frac{1}{2h}u_k,(x_1^*,x_2^*)\frac{1}{2h}(x_1,x_2),\ t^*=\frac{c_s}{2h}t,\ q^*=\frac{1}{G}q,\ h^*=\frac{1}{2},\ c_s^2=\frac{G}{\rho}.$$

When investigating the propagation of waves, dimensionless quantities are introduced: $t^*=\dfrac{1}{2h}t$ is the wavelength, $c^*=\dfrac{c}{c_s}$ is the phase velocity.

The generalized equation

The generalized differential equation with respect to the transverse coordinate (the asterisks are omitted) is of the form [1]

$$\left\{\left[\left(\frac{\partial^2}{\partial t^2}+a_1\nabla^2\nabla^2\right)_K-a_2\frac{\partial^2}{\partial t^2}\nabla^2+a_3\frac{\partial^4}{\partial t^4}\right]_{TM}-\right.$$
$$\left.-b_1\nabla^2\nabla^2\nabla^2+b_2\frac{\partial^2}{\partial t^2}\nabla^2\nabla^2-b_3\frac{\partial^4}{\partial t^4}\nabla^2+b_4\frac{\partial^6}{\partial t^6}\right\}_{TMC}w_0= \qquad (7)$$
$$=\left\{\left[1-d_1\nabla^2+d_2\frac{\partial^2}{\partial t^2}\right]_{TM}+d_3\nabla^2\nabla^2-d_4\frac{\partial^2}{\partial t^2}\nabla^2+d_5\frac{\partial^4}{\partial t^4}\right\}_{TMC}(q^+-q^-).$$

This equation has been derived from n-dimensional Euclidean space. In (7) the following notations are accepted: $w_0(\vec{x},t)$ is the transverse displacement (deflection), t is the time, $(q_1-q_2)$is the lateral load, coefficients depend on the Poisson ration only.

The operator with index K corresponds to the Bernoulli-Euler equation (extended to plates by Kirchhoff). The operator with the TM index corresponds to the Timoshenko equation [3]. It has been extended to the plates by Uflyand

and developed by Mindlin. The Rayleigh equation is a part of the operator TM at a_3 = 0. The operator with the TMC index corresponds to the generalized equation. The Timoshenko equation follows from the above analytical construction as a special case, but without introducing a correction parameter (thickness-shear).

Timoshenko [3] has been the first who generalized the parabolic equation for the propagation of flexural vibrations of a beam to a hyperbolic equation, applying the phenomenological approach, so that the normal does not remain normal to the middle surface under bending deformations of the beam, which is not taken into account by the model continuous medium. In the classical models of Bernoulli-Euler, Kirchhoff, Rayleigh the normal remains normal. The effects of Timoshenko appear locally in the presence of sharp inhomogeneities, in wave theory these are short waves. In this case, it is necessary to apply a more general theory than the classical one, for example, the Cossera theory [9].

Comparison of models

It is impossible to change the type of the differential operator by any correction factor, it is possible only to approximate the description to the model predicted by the continuous medium by selecting this coefficient.

In the classical theory of continuous media, it is assumed that the forces acting on an infinitesimal element are collinear forces in spite of bending. In the presence of a moment field, this field can be regarded as appearing due to the noncollinear forces θ [9] assumed the collinearity of forces and introduced the moment θ. However, in any case, this can be understood as a discontinuity. This is in accordance with Timoshenko's equation, when the normal to the middle surface does not remain normal after deformation.

The equations of the Cosserat theory are introduced. Without loss of generality, we can obtain a plane problem by rotating a vertical axis Ox_1. In accordance with the dimensionless quantities introduced above and supplemented by the Cosserat parameters (the asterisks are further omitted) there are $\alpha, \beta, \gamma, \varepsilon_k, f$. The equations for the displacement vector $\vec{u}$ and the rotation vector $\vec{\theta}$ are

$$\nabla^2\vec{u} + (1+\lambda/G)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \alpha\vec{\nabla}\times\vec{\nabla}\times\vec{u} + 2\alpha\vec{\nabla}\times\vec{\theta} = \partial_{tt}\vec{u}\,, \qquad (8)$$

$$(\beta+2\gamma)\vec{\nabla}(\vec{\nabla}\cdot\vec{\theta}) - (\gamma+\varepsilon_k)\vec{\nabla}\times\vec{\nabla}\times\vec{\theta} + 2\alpha\vec{\nabla}\times\vec{u} - 4\alpha\vec{\theta} = j\partial_{tt}\vec{\theta}. \qquad (9)$$

Thus, the introduction of moment stresses can be understood as the action of noncentral forces, and this is a violation of continuity, and when the beam is bent, the normal does not remain a normal to the deformed middle surface. This situation is essentially postulated in the derivation of the Timoshenko equation: the slope of the tangent to the bending curve $\partial w/\partial x = \psi + \gamma$ is represented in the form, where ψ is the bending deformation, γ is the shear deformation. At high frequencies or sharp inhomogeneities this will manifest itself.

From the point of view of the differential operators theory, Timoshenko generalization is essentially nontrivial, since in this case a parabolic operator of a higher order (the fourth, not the second) is generalized, in contrast to all previous generalizations (diffusion, heat, etc.).

Violations of continuity and some statements on incorrection

Extensive attempts for adapting the Timoshenko hyperbolic equation to the classical parabolic equation were conducted. In most studies, variational formulations and asymptotic approaches of the Timoshenko model using the law of continuity of the medium show the incorrectness of the Timoshenko model. Therefore, all further arguments and conclusions about the frequency spectra and the meaning of the second spectrum remain in question (Barbashov, Nesterenko, Chervyakov).

An attempt to use the mathematical method of asymptotic expansions taking into account the continuity of the medium leads to the incorrectness estimations of the Timoshenko equation (Bakhvalov and Eglit, 2005) [10].

Hutchinson published the paper about adapting the correcting shear coefficient determining from exact solutions of the problems based on classical continuum approach [11].

Some properties of differential operators

For clarity, the equations describing the propagation of one-dimensional waves, which follow from (7) under rotation with respect to the vertical axis normal to the middle surface are presented:

The Bernoulli-Euler equation was extended to plates by Kirchhoff

$$\frac{\partial^2 w}{\partial t^2}+\frac{D}{\rho h}\frac{\partial^4 w}{\partial x^4}=0, \qquad (10)$$

The Rayleigh equation takes into account the inertia of rotation

$$\frac{\partial^2 w}{\partial t^2}+\frac{D}{\rho h}\frac{\partial^4 w}{\partial x^4}-\left(\frac{I}{h}\right)\frac{\partial^4 w}{\partial t^2 \partial x^2}=0, \qquad (11)$$

The Timoshenko equation takes into account the thickness-shear

$$\frac{\partial^2 w}{\partial t^2}+\frac{D}{\rho h}\frac{\partial^4 w}{\partial x^4}-\left(\frac{D}{k_s^2 Gh}+\frac{I}{h}\right)\frac{\partial^4 w}{\partial t^2 \partial x^2}+\frac{\rho I}{k_s^2 Gh}\frac{\partial^4 w}{\partial t^4}=0 , \qquad (12)$$

Equation (12), including (10) and (11), was presented in the substantial article by Timoshenko in 1921 [3] and was widely distributed as the Timoshenko equation.

Separation of variables in operators

We also note the fundamental difference between the Rayleigh equation including the Euler-Bernoulli equation,

$$\xi\frac{\partial^2 w}{\partial t^2}+\xi^3 a_1\frac{\partial^4 w}{\partial x^4}-\xi^3 a_2\frac{\partial^4 w}{\partial t^2\partial x^2}=\left(q^+-q^-\right), \tag{13}$$

and the Timoshenko equation

$$\xi\frac{\partial^2 w}{\partial t^2}+\xi^3 a_1\frac{\partial^4 w}{\partial x^4}-\xi^3 a_2\frac{\partial^4 w}{\partial t^2\partial x^2}+\xi^3 a_3\frac{\partial^4 w}{\partial t^4}=$$
$$=\left(1-\xi^2 d_1\frac{\partial^2}{\partial x^2}+\xi^2 d_2\frac{\partial^2}{\partial t^2}\right)\left(q^+-q^-\right). \tag{14}$$

A classical method of the variable separation

$$w\left(x,t\right)=W\left(x\right)T\left(t\right) \tag{15}$$

does not lead to the separation of variables (15) in equation (14), in contrast to the complete separation of variables in equation (13). In the case of harmonic oscillations, the method is applicable to both (13) and (14).

Conclusion

The Timoshenko equation singularity has been established for studying the violation effect on wave propagation. The finite speed of disturbance propagation has been studied for considering this effect on wave propagation especially for short wavelengths and sharp changes of the beam cross- sections unlike many works before. The Timoshenko model has been compared with the Cosserat equations for obtaining the continuity violation of elastic medium. Error statements of some researchers about incorrectness of Timoshenko equation are noted. Some features of Timoshenko equation are presented.

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